INFINITESIMAL CASTELNUOVO THEORY IN ABELIAN VARIETIES

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Abstract. The purpose of this article is to show that the Castelnuovo theory for abelian varieties, developed by G. Pareschi and M. Popa, can be infinitesimalized. More precisely, we prove that an irreducible principally polarized abelian variety has a finite scheme in extremal position, in the sense of Castelnuovo theory for abelian varieties, if, and only if, it is a Jacobian and the scheme is contained in a unique Abel-Jacobi curve.

1. Introduction

The classical Castelnuovo theory on the projective space deals with the conditions imposed on quadrics by a finite number of points. In particular, consider a collection of $2n+3$ points in linearly independent position in a projective space $\mathbb{P}^n$. Then, it imposes $2n+1$ conditions on quadrics if, and only if, it is contained in a unique rational normal curve.

D. Eisenbud and J. Harris proved in [EH] that this result can be infinitesimalized. More precisely, they define that a finite subscheme is in linearly general position if for each proper linear space $\Lambda \subset \mathbb{P}^n$, we have $\deg(\Lambda \cap \Gamma) \leq 1 + \dim \Lambda$. Next, they prove that if Γ is a finite scheme of $\mathbb{P}^n$ in linearly general position and $\deg \Gamma \geq 2n+3$, then there is a unique rational normal curve that contains Γ (cf. [EH, Theorem 4]).

On the other hand, G. Pareschi and M. Popa have given an abelian analogue of Castelnuovo theory (cf. [PP2]) that surprisingly characterizes Jacobians among principally polarized abelian varieties (ppav's for short). Namely, they define that $d \geq g+1$ points in a ppav (A, Θ) of dimension g are in theta-general position if, for any subset of $g+1$, there exists a translate of Θ that contains g of these points and avoids the remaining one. Then an irreducible ppav of dimension g, that contains a $g+2$ points in theta-general position, imposing only $g+1$ conditions on the linear series $|\mathcal{O}_A(2\Theta) \otimes \alpha|$ for α general is the Jacobian of a curve C. Moreover, the points lie on a unique Abel-Jacobi curve (cf. [PP2]). We follow the approach of G. Pareschi and M. Popa, but we observe that a related result was obtained independently by S. Grushevsky (see [Gr1, Gr2]).

Similarly to the extension that D. Eisenbud and J. Harris did in the case of Castelnuovo theory for projective spaces, we extend the definition of theta-general position to the case of nonreduced finite schemes. We say that a finite scheme of degree $d \geq g+1$ in a ppav (A, Θ) of dimension g is in theta-general position if for any two subschemes $\Gamma'' \subset \Gamma'$ with $\deg \Gamma'' + 1 = \deg \Gamma' \leq g+1$, there is a translate of Θ that contains Γ'' but do not contain Γ'.

The previous definition generalizes the former of G. Pareschi and M. Popa, and it is "good" in the sense that theta-general finite schemes behave analogously to the case of distinct points. For instance, we have:

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- A general finite scheme of degree $d \geq g+1$ on an Abel-Jacobi curve is in theta-general position, by the Jacobi inversion theorem.
- A finite scheme in theta-general position and degree $d \geq g+1$ imposes at least $g+1$ conditions on the general linear series $|\mathcal{O}_A(2\Theta) \otimes \alpha|$ (cf. Proposition 3.8).

Now, following G. Pareschi and M. Popa, we observe that a finite scheme of degree $d \geq g+1$ on an Abel-Jacobi curve, imposes precisely $g+1$ conditions on the linear series $|\mathcal{O}_A(2\Theta) \otimes \alpha|$ for $\alpha \in J(C)$ general. Indeed, this is an easy consequence of the vanishing of $H^1(\mathcal{I}_C(2\Theta) \otimes \alpha)$ for a general $\alpha \in \widehat{A}$ (cf. [PP1]). Therefore, a general finite scheme on an Abel-Jacobi curve impose the smallest possible number of conditions on general "abelian-quadrics".

In this paper we prove that the converse is also true. More precisely, the minimality on the conditions imposed on "abelian-quadrics" gives a geometric criterion to identify Jacobian varieties. Moreover, the finite schemes imposing these minimal conditions have to lie on an unique Abel-Jacobi curve.

The following result is the main theorem of the paper. In the case of a reduced finite subscheme (distinct points), it was already proved by G. Pareschi and M. Popa (cf. main theorem of [PP2] and [Gr1, Gr2]).

Theorem A. *Let (A, Θ) be an irreducible principally polarized abelian variety of dimension $g > 3$, and let $\Gamma \subset A$ be a finite scheme of degree $d \geq g+2$, such that it is in theta-general position, but imposes only $g+1$ conditions on the linear series $|\mathcal{O}_A(2\Theta) \otimes \alpha|$ for α general in A. Then (A, Θ) is the canonically polarized Jacobian of a curve C and $\Gamma \subset C$ for a unique Abel-Jacobi embedding $C \hookrightarrow J(C)$.*

So we give a Schottky-type criterion, namely a geometric condition that characterizes Jacobians among all ppav's. We require that $g > 3$, but note that any ppav of dimension less or equal than 3 is a Jacobian variety or a product of Jacobian varieties.

We also observe that this theorem implies an abelian analogue of [EH, Theorem 1(a)]:

Corollary B. *Any finite scheme $\Gamma \subset A$ of degree $d \geq g+2$, such that it is in theta-general position, but imposes only $g+1$ conditions on the linear series $|\mathcal{O}_A(2\Theta) \otimes \alpha|$ for α general in A, lies on a smooth curve, so it is composed by curvilinear points.*

We will focus our attention to the case of a finite scheme supported in a unique point, because all the difficulties appear in this case. Roughly speaking, the key points in the proof of the Theorem are similar to those of G. Pareschi and M. Popa. Namely, for any subscheme of degree $g+1$, we consider the locus V of $a \in A$ where the subscheme fails to impose independent conditions on $|\Theta + t_a^*\Theta|$. The divisorial part of V is a translate of Θ. We can describe the precise translation in terms of the subscheme of degree $g+1$ (cf. Lemma 5.2). Then, new difficulties arise to find a unidimensional family of trisecants. We overcome them proving that the intersection of the divisorial part of V with a unidimensional family of translates of Θ is nonreduced. More precisely, we prove that it is included in the union of two translates of Θ_{Γ_2}, which is the locus of $a \in A$ such that $t_a^*\Theta$ contains a finite scheme Γ_2 of degree 2 (cf. Lemma 6.4). This way, we obtain a positive-dimensional family of degenerate trisecants to the Kummer variety (cf. Proposition 6.7) using the work of O. Debarre (see [De]) and G. Marini (see [Ma]). Hence, we can apply the Gunning-Welters criterion (cf. [W2]) which implies that (A, Θ) is a Jacobian. In section § 7, we sketch how we can adapt the proof for a finite scheme supported in a unique point to prove the general case.

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2. Theta Containing Locus

If (A,Θ) is a ppav we note Θ_p the translation $t_p^*\Theta$, where $t_p(x)=x+p$. Then if Θ is a symmetric theta divisor we have that $p\in\Theta_q$ if, and only if, $q\in\Theta_p$. So Θ_p is the locus of $q\in A$ such that $p\in\Theta_q$.

For a finite scheme Γ we define

$$\Theta_\Gamma := \{\alpha\in A \mid \Gamma\subset\Theta_\alpha\}\,.$$

Note 2.1. We observe that $\dim\Theta_\Gamma\geq\dim A-\deg\Gamma$. More precisely, every component has dimension greater or equal that $\dim A-\deg\Gamma$.

Example 2.2. Given a finite scheme Γ consisting in k distinct points $\{p_1,\dots,p_k\}$, $\Theta_\Gamma=\Theta_{p_1}\cap\cdots\cap\Theta_{p_k}$.

Example 2.3. We want to study the case where Γ is a subscheme of degree 2 and supported in one unique point. In such a situation Γ can be seen as giving a tangent vector v in one point $p\in A$, i.e., $v\in T_A(p)$. Then consider the Gauss map

$$\begin{array}{rcl}\pi:\Theta_p-\mathrm{Sing}(\Theta_p) & \longrightarrow & \mathbb{P}^\vee=\mathbb{P}(T_A(0))^\vee\\ q & \longmapsto & t_{-q}^*\left(\frac{\partial\vartheta_p}{\partial z_1}(q):\cdots:\frac{\partial\vartheta_p}{\partial z_g}(q)\right)\end{array},$$

where ϑ is a local equation of Θ_p on q.

As the tangent bundle $\mathcal{T}_A$ is a trivial sheaf, v is a vector in $T_A(p)\overset{t^*}{\cong}T_A(0)$ and it determines an hyperplane $H_v\in\mathbb{P}^\vee$. So we can define

$$\Theta_\Gamma := \left(\overline{\pi^*H_v}\right)_{\mathrm{red}}$$

Then we can prove the result we are interested in

Claim: $\Gamma\subset\Theta_q\Leftrightarrow q\in\Theta_\Gamma$.

Proof. We will consider any tangent space of a theta translate as a subspace of the tangent vector of the abelian variety at the origin via the canonical translation, i.e.

$$T_{\Theta_x}(y)\overset{t_{-y}^*}{\cong}T_{\Theta_{x-y}}(0)\subset T_A(0)$$

Then as we have chosen a symmetric theta divisor Θ,

$$T_{\Theta_x}(y)=(-1)^*T_{\Theta_y}(x)\text{ as a subspaces of }T_A(0), \tag{1}$$

Hence,

$\boxed{\Leftarrow}$ If $q\in\Theta_\Gamma$ then $q\in\Theta_p$ and hence $p\in\Theta_q$. Moreover
- if $q\in\pi^{-1}H_v$ then $\pi(q)\in H_v$, so $t_{-q}^*\left(\frac{\partial\vartheta_p}{\partial z_1}(q):\cdots:\frac{\partial\vartheta_p}{\partial z_g}(q)\right)\in H_v$ and by (1) $\pm v\in T_{\Theta_p}(q)\Rightarrow\mp v\in T_{\Theta_q}(p)$,

- otherwise $q \in \mathrm{Sing}(\Theta_p)$ and then, as $T_{\Theta_p}(q) \overset{t^*_{-q}}{\cong} T_{\Theta_{p-q}} = T_A(0)$, we have that $\pm v \in T_A(0) \Rightarrow \mp v \in T_{\Theta_q}(p)$.

$\boxed{\Rightarrow}$ If $\Gamma \subset \Theta_q$ then $p \in \Theta_q$ so $q \in \Theta_p$ and moreover

- if q is a nonsingular point of Θ_p, then $\pm v \in T_{\Theta_q}(p) \Rightarrow \mp v \in T_{\Theta_p}(q) \Rightarrow q \in \pi^{-1}H_v$.
- otherwise $q \in \mathrm{Sing}(\Theta_p)$ and then, $T_{\Theta_q}(p) = T_A(p)$. Hence $q \in \pi^{-1}H_v$.

So $q \in \Theta_\Gamma$.

We observe that in fact, $H_v = H_{-v}$ □

Remark 2.4. We observe that given Γ a reduced scheme of degree 2 supported at 0, if $v \in T_A(0)$ is the associated tangent vector (i.e. $T_\Gamma(0) = \langle v \rangle$) we can consider $\overline{\theta}_v \in H^0\mathcal{O}_\Theta(\Theta)$, the image of $v \in H^0T_A$ by the isomorphism $H^0T_A \to H^0\mathcal{O}_\Theta(\Theta)$ describing the first order infinitesimal variations of Θ in A (this isomorphism being determined by the choice of an equation $\theta \in H^0\mathcal{O}_A(\Theta)$ for Θ). Then,

$$\Theta_\Gamma = \left(\overline{\Theta}_{\langle v \rangle}\right)_{\mathrm{red}},$$

where $\overline{\Theta}_{\langle v \rangle}$ is the divisor defined by $\overline{\theta}_v \in H^0\mathcal{O}_\Theta(\Theta)$.
Another equivalent way to define Θ_Γ is to consider the invariant vector field $D \neq 0$ associated to Γ and then,

$$\Theta_\Gamma = (D\Theta)_{\mathrm{red}},$$

where $D\Theta$ is the scheme of zeroes of the section $D\theta \in H^0(\Theta, \mathcal{O}_\Theta(\Theta))$. In fact, $D\theta = \overline{\theta}_{\langle v \rangle}$

3. Theta-general Position

Recall that following Pareschi and Popa [PP2, Definition 3.1], a collection Z of $n \leq g+1$ distinct points on A is theta-independent if, for any decomposition on Z as $Z = Y \cup \{p\}$, there is a theta translate Θ_γ such that $Y \subset \Theta_\gamma$ and $p \notin \Theta_\gamma$.

We generalize the notion of theta-independence to a possibly reduced scheme.

Definition 3.1. A 0-dimensional subscheme $\Gamma \subseteq A$ of degree $d \leq g+1$ is theta-independent if for every composition series

$$\varnothing = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \ldots \subsetneq \Gamma_{d-1} \subsetneq \Gamma_d$$

and for all $i = 1, \ldots, d$, there is a theta translate Θ_α such that $\Gamma_{i-1} \subset \Theta_\alpha$ and $\Gamma_i \not\subset \Theta_\alpha$.

Remark 3.2. This definition is a generalization of [PP2, Definition 3.1] because if Γ is a reduced 0-scheme they coincide. Indeed, if the reduced 0-scheme Γ is theta-independent in the sense of [PP2, Definition 3.1], consider a pair of subschemes $\Gamma'' \subset \Gamma' \subseteq \Gamma$ such that $\deg \Gamma'' + 1 = \deg \Gamma'$ then we can take $\Gamma'' \cup (\Gamma - \Gamma')$ that are $\deg \Gamma - 1$ distinct points. By the theta-independence of Γ there exists an $\alpha \in A$ such that,

$$\left.\begin{array}{l} \Gamma'' \cup (\Gamma - \Gamma') \subset \Theta_\alpha \\ \Gamma \not\subset \Theta_\alpha \end{array}\right\} \Rightarrow \begin{array}{l} \Gamma'' \subset \Theta_\alpha \\ \Gamma' \not\subset \Theta_\alpha. \end{array}$$

In the other sense, the statement is obvious.

Definition 3.3. A 0-dimensional subscheme $\Gamma \subseteq A$ of degree $d \geq g+1$ is in theta-general position if, for all subscheme $\Gamma' \subseteq \Gamma$ of degree $g+1$, Γ' is theta-independent.

Note 3.4. Note that, if a scheme has depth $d > g+1$ in some point, the infinitesimal points in depth greater than $g+1$ are irrelevant when checking the theta-general condition.

3.1. **Basic Lemma.** In order to characterize the divisors linearly equivalent to 2Θ that pass through some specific finite scheme we need the following definition,

Definition 3.5. If $\Gamma' \subset \Gamma$ is a subscheme such that $\deg \Gamma' + 1 = \deg \Gamma$, we denote by $z_{\Gamma',\Gamma}$ the support of the irreducible component of Γ that is not contained in Γ'.
We observe that $z_{\Gamma',\Gamma}$ is a reduced unique point.

Then Nakayama lemma provides us the next result.

Lemma 3.6. *Let Γ be a finite subscheme of an algebraic variety X and let $\Gamma' \subset \Gamma$ be a (finite) subscheme such that*

$$\deg \Gamma' + 1 = \deg \Gamma$$

Then, if D is a divisor of X such that $\Gamma' \subset D$ but $\Gamma \not\subset D$ we get

$$\{E \text{ divisor of } X \mid \Gamma \subset D + E\} = \big\{E \text{ divisor of } X \mid z_{\Gamma',\Gamma} \in E\big\}$$

Proof. As $\Gamma' \subset \Gamma$ differ only in one degree we can consider that they are supported in one unique point p.

$\boxed{\subseteq}$ From $\Gamma' \subset \Gamma$ we consider the corresponding ideals $\mathcal{I}' \supset \mathcal{I}$ that by hypothesis are of codimension 1, and the maximal $\mathfrak{m}$ corresponding to p. In such a local situation the divisor D corresponds to an element $f \in \mathcal{I}' - \mathcal{I}$.
The quotient $\mathcal{I}'/\mathcal{I}$ has dimension 1, hence,
- if $\mathfrak{m}\,(\mathcal{I}'/\mathcal{I}) = \mathcal{I}'/\mathcal{I}$ then by Nakayama lemma $\mathfrak{m}\,(\mathcal{I}'/\mathcal{I}) = 0$,
- otherwise $\mathfrak{m}\,(\mathcal{I}'/\mathcal{I}) \subsetneq \mathcal{I}'/\mathcal{I}$ and, as $\mathcal{I}'/\mathcal{I}$ has dimension 1, we also have that $\mathfrak{m}\,(\mathcal{I}'/\mathcal{I}) = 0$.

So, in any case, $\mathfrak{m}\mathcal{I}' \subset \mathcal{I}$. Then if $g \in \mathfrak{m}$, $f \cdot g \in \mathcal{I}$, in other words, if $z_{\Gamma',\Gamma} \in E$ then $\Gamma \subset D + E$.

$\boxed{\supseteq}$ Obvious because if $g \notin \mathfrak{m}$, then g is a unit and, as $f \notin \mathcal{I}$, $f \cdot g \notin \mathcal{I}$.

□

Following [EH] a finite scheme Γ is a curvilinear point if its is supported in one unique point x_0 and the local ring $\mathcal{O}_{\Gamma,x_0} \cong \mathbb{C}[x]/(x^d)$. Then we can extend the previous result to the following situation:

Lemma 3.7. *Let Γ be curvilinear point of degree d in an algebraic variety X and let*

$$\varnothing = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \ldots \subsetneq \Gamma_{d-1} \subsetneq \Gamma_d = \Gamma,$$

be its unique composition series. Then, if D is a divisor of X such that $\Gamma_i \subset D$ but $\Gamma_{i+1} \not\subset D$ we get

$$\{E \text{ divisor of } X \mid \Gamma_{i+j} \subset D + E\} = \{E \text{ divisor of } X \mid \Gamma_j \in E\}$$

Proof. As Γ is a curvilinear point, if p is the support of Γ, $\mathcal{O}_{\Gamma,p} \cong \mathbb{C}[x]/(x^d)$ and we can reduce the pass condition of a divisor by a subscheme Γ_i only checking the belonging of its associated local function to the principal ideal (x^i). Consider f and g are local functions of D and E in p. If we assume that $f \in (x^i) - (x^{i+1})$, then $f \cdot g \in (x^{i+j})$, if, and only if, $g \in (x^j)$. □

3.2. **Castelnuovo Lemma.** If (A, Θ) is a principally polarized abelian variety, the isogeny

$$\begin{array}{rcl} \Phi : (A,\Theta) & \longrightarrow & (\widehat{A}, \widehat{\Theta}) \\ p & \longmapsto & \mathcal{O}_A(\Theta_p - \Theta) \end{array}$$

is, indeed, an identification. Then the translates of theta, Θ_p, can be seen as the divisors associated to $\mathcal{O}_A(\Theta) \otimes \alpha$ where $\alpha \in Pic^0(A) \cong \widehat{A}$ is the element associated by the identification Ψ. Moreover this identification allows us to notate

$$\mathcal{O}_A(k\Theta)_\alpha := \mathcal{O}_A(k\Theta) \otimes \alpha, \qquad \text{where } \alpha \in Pic^0(A)$$

and think it as a *theta-translate*. Abusing notation, we will often consider $\mathcal{O}_A(k\Theta)_\alpha$, where $\alpha \in A$, or also the *divisor* $(2\Theta)_\alpha$.

Proposition 3.8. *Let $\Gamma \subset A$ 0-dimensional scheme in theta-general position. Then Γ imposes at least $\min\{\deg\Gamma, g+1\}$ independent conditions on $H^0(\mathcal{O}(2\Theta)_\alpha)$ for a general $\alpha \in A$.*

Proof. Let be $d = \min\{\deg\Gamma, g+1\}$ and let

$$\varnothing = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \ldots \subsetneq \Gamma_{d-1} \subsetneq \Gamma_d \subseteq \Gamma$$

a composition series for Γ_d a subscheme of degree d in Γ. As $\Gamma_d \subseteq \Gamma$ is theta-independent, there exists $\alpha_{\{i-1,i\}}$ (has not to be unique) such that $\Theta_{\alpha_{\{i-1,i\}}} \supset \Gamma_{i-1}$ but $\Theta_{\alpha_{\{i-1,i\}}} \not\supset \Gamma_i$. As $\Gamma_i \subset \Theta_{\alpha_{\{i-1,i\}}} + \Theta_\beta$ if, and only if, $z_{\Gamma_{i-1},\Gamma_i} \in \Theta_\beta$ (cf. 3.6), then $\beta \in \Theta_{z_{\Gamma_{i-1},\Gamma_i}}$. If we call $\alpha_{\{i-1,i\}} + \beta = \alpha$ we have showed that for each α such that $\alpha - \alpha_{\{i-1,i\}} \notin \Theta_{z_{\Gamma_{i-1},\Gamma_i}}$, there is a divisor $\Theta_{\alpha_{\{i-1,i\}}} + \Theta_{\alpha-\alpha_{\{i-1,i\}}} \in |(2\Theta)_\alpha|$ that contains Γ_{i-1} and does not contain Γ_i.
Thus, the theta-generality provides to us a set of d divisors in $|(2\Theta)_\alpha|$ which are all independent, because they vanish differently when evaluated on Γ. Therefore, if

$$\alpha \notin \bigcup_i \Theta_{z_{\Gamma_{i-1},\Gamma_i}+\alpha_{\{i-1,i\}}},$$

then Γ imposes at least d conditions on $H^0(\mathcal{O}_A(2\Theta)_\alpha)$. □

Definition 3.9. If $\Gamma \subseteq A$ is a 0-dimensional scheme, for each subscheme $\Gamma' \subseteq \Gamma$ such that $\deg\Gamma' + 1 = \deg\Gamma$ we define

$$\mathcal{H}^{\Gamma',\Gamma} = \left\{\alpha \in A \mid \Gamma' \subset \Theta_\alpha \text{ but } \Gamma \not\subset \Theta_\alpha\right\}$$

Remark 3.10. We observe that

$$\mathcal{H}^{\Gamma',\Gamma} = \Theta_{\Gamma'} - \Theta_\Gamma$$

and if Γ is a theta-independent scheme, then Θ_Γ is a proper closed set in $\Theta_{\Gamma'}$, so the closure of $\mathcal{H}^{\Gamma',\Gamma}$ is the union of some components of $\Theta_{\Gamma'}$. Therefore (cf. note 2.1), $\dim\mathcal{H}^{\Gamma',\Gamma} \geq g - \deg\Gamma + 1$ (the *expected* dimension).

4. Cohomological Support Locus

Definition 4.1 ([PP2, Definition 4.4])**.** Let Γ be a finite scheme on A. We consider the cohomological support locus

$$V(\mathcal{I}_\Gamma(k\Theta)) := \left\{\alpha \in A \mid h^1(\mathcal{I}_\Gamma((k\Theta)_\alpha) \geq 1\right\}.$$

As G. Pareschi and M. Popa observe, since $h^i(\mathcal{O}_A((k\Theta)_\alpha)) = 0$ for any $i > 0$ and any $\alpha \in A$, $V(\mathcal{I}_\Gamma(k\Theta))$ is the locus of $\alpha \in A$ such that Γ fails to impose independent conditions on $|(k\Theta)_\alpha|$.

Then, in terms of this definition, Proposition 3.8 says that for any theta-general finite scheme Γ of degree at most $g+1$, the cohomological support locus $V(\mathcal{I}_\Gamma(2\Theta))$ is a proper subvariety.

Definition 4.2. Let Γ be a finite scheme in A and let $\Gamma'' \subset \Gamma' \subseteq \Gamma$ be any two subschemes of Γ such that $\deg\Gamma'' + 1 = \deg\Gamma'$. We denote

$$B(\Gamma'' \subseteq \Gamma', k) := \left\{\alpha \in A \mid H^0(I_{\Gamma'}(k\Theta)_\alpha) = H^0(I_{\Gamma''}(k\Theta)_\alpha)\right\}$$

The previous definitions relate each other by the following equality,

$$V(I_\Gamma(k\Theta)) = B(\Gamma' \subseteq \Gamma, k) \cup V(I_{\Gamma'}(k\Theta)) \tag{2}$$

In other words, if Γ fails to impose independent conditions on $|(k\Theta)_\alpha|$, then either a subscheme Γ' of codegree 1 imposes the same conditions as Γ, or Γ' also fails to impose independent conditions.

4.1. **Dimension Bounds.** To prove the inclusions that allows us to find degenerate trisecants, it is useful to control the dimensions of the intersections parameterized by the sets $\mathcal{H}^{\Gamma',\Gamma}$. The next lemma is a generalization of [PP2, Lemma 3.6].

Lemma 4.3. *Let* Γ *be a finite theta-independent scheme of degree* $d \leq g+1$*. Then, for any subscheme* $\Gamma' \subset \Gamma$ *such that* $\deg \Gamma' + 1 = \deg \Gamma$*,*

$$\dim\Big(\bigcap_{\alpha \in \mathcal{H}^{\Gamma',\Gamma}} \Theta_\alpha \Big) \leq d-2.$$

Proof. Done by descending induction with respect to d:

- For $d = g+1$ the assertion is trivial, because as Γ is theta-independent $\mathcal{H}^{\Gamma',\Gamma} \neq \varnothing$.
- If $d < g+1$, let x be a sufficiently general point (in particular, let $\Gamma \cup \{x\}$ be a theta-independent scheme) and denote $W = \bigcap_{\alpha \in \mathcal{H}^{\Gamma' \cup \{x\}, \Gamma \cup \{x\}}} \Theta_\alpha$. By induction hypothesis $\dim W \leq d-1$ and we have

$$\dim\Big(\bigcap_{\alpha \in \mathcal{H}^{\Gamma',\Gamma}} \Theta_\alpha \Big) = \dim\Big(W \cap \Big(\underbrace{\bigcap_{\alpha \in \left(\mathcal{H}^{\Gamma',\Gamma} - \mathcal{H}^{\Gamma' \cup \{x\}, \Gamma \cup \{x\}}\right)} \Theta_\alpha}_{B} \Big)\Big)$$

  We suppose W irreducible of dimension $d-1$ (if not, consider an irreducible component of W of $\dim d-1$).

  Claim: $\forall\, X \subset A$ irreducible of dimension $r \geq 1$, $\exists\, p_1, \ldots, p_{r+1} \in X$ such that

$$\dim \Theta_{p_1} \cap \cdots \cap \Theta_{p_{r+1}} = g - (r+1)$$

  Then we argue by contradiction.
  If $\dim W \cap B = d-1$, then $W \subseteq \bigcap_{\alpha \in \left(\mathcal{H}^{\Gamma',\Gamma} - \mathcal{H}^{\Gamma' \cup \{x\}, \Gamma \cup \{x\}}\right)} \Theta_\alpha$ and we choose $p_1, \ldots, p_n$ points in W verifying the claim:

$$\dim \Theta_{p_1} \cap \cdots \cap \Theta_{p_d} = g - d$$

  then

$$p_i \in W \subseteq \bigcap_{\alpha \in \left(\mathcal{H}^{\Gamma',\Gamma} - \mathcal{H}^{\Gamma' \cup \{x\}, \Gamma \cup \{x\}}\right)} \Theta_\alpha \Rightarrow p_i \in \Theta_\alpha \;\forall\, \alpha \in \mathcal{H}^{\Gamma',\Gamma} - \mathcal{H}^{\Gamma' \cup \{x\}, \Gamma \cup \{x\}}$$

$$\Rightarrow \mathcal{H}^{\Gamma',\Gamma} - \mathcal{H}^{\Gamma' \cup \{x\}, \Gamma \cup \{x\}} \subseteq \Theta_{p_1} \cap \cdots \cap \Theta_{p_d}$$

  then as all the components of $\mathcal{H}^{\Gamma',\Gamma}$ have $\dim \geq g-d+1$ (cf. 3.10) we get a contradiction with $\dim \Theta_{p_1} \cap \cdots \cap \Theta_{p_d} = g-d$, i.e. if C is a component of $\mathcal{H}^{\Gamma',\Gamma}$ non contained in $\mathcal{H}^{\Gamma' \cup \{x\}, \Gamma \cup \{x\}}$ then

$$g - d + 1 \leq \dim C \leq g - d \text{ !!!}$$

$\square$

Proof of the claim.

$$\begin{tikzcd}[column sep=small]
 & X \times \overset{(r+1)}{\cdots} \times X \times A & \\
 & I = \left\{ (p_1, \ldots, p_{r+1}, \alpha) \mid \alpha \in \Theta_{p_1} \cap \ldots \cap \Theta_{p_{r+1}} \right\} \arrow[u, hook] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X \times \overset{(r+1)}{\cdots} \times X & & A
\end{tikzcd}$$

We apply the theorem of the fiber dimensions:

- π_2 is surjective.
- For a generic α, $\dim \pi_2^{-1}(\alpha) = (r-1)(r+1) = r^2 - 1$. Then $\dim I = g + r^2 - 1$.
- π_1 is surjective if $r + 1 \leq g$.

Then, in the generic case

$$\begin{aligned}\dim \Theta_{p_1} \cap \ldots \cap \Theta_{p_{r+1}} &= \dim \pi_1^{-1}(p_1, \ldots, p_{r+1})\\ &= \dim I - \dim X^{r+1}\\ &= g - (r+1)\end{aligned}$$

□

Now we can prove the infinitesimal version of [PP2, Lemma 4.7] that bounds de dimension of the cohomological support locus $V(\mathcal{I}_\Gamma(2\Theta))$.

Lemma 4.4. *Let Γ be a theta-independent scheme of degree $d \leq g+1$. Then* $\dim V(\mathcal{I}_\Gamma(2\Theta)) \leq d - 2$.

Proof. we prove it by induction on the degree of Γ. Take a subscheme $\Gamma' \subset \Gamma$ such that $\deg \Gamma' + 1 = \deg \Gamma$. Then for any $\gamma \in \mathcal{H}^{\Gamma',\Gamma}$ and for any theta-translate Θ_α that do no contain $z := z_{\Gamma',\Gamma}$, the divisor $\Theta_\gamma + \Theta_\alpha$ contains Γ' and do not contain Γ (cf. 3.6). Hence, the fixed points between Γ' and Γ must be contained in $\Theta_{z+\gamma}$ for all $\gamma \in \mathcal{H}^{\Gamma',\Gamma}$, so $B(\Gamma' \subset \Gamma, 2) \subset \bigcap_{\gamma \in \mathcal{H}^{\Gamma',\Gamma}} \Theta_{z+\gamma}$. Recall that the previous Lemma 4.3 assures us that $\dim B(\Gamma' \subset \Gamma, 2) \leq d - 2$. Using the equality (2) and the induction hypothesis, $\dim V(\mathcal{I}_\Gamma(2\Theta)) \leq d - 2$. □

5. Extremal Position

From now on (A, Θ) will be assumed to be an *irreducible* ppav (i.e. Θ is irreducible) of dimension $g > 3$.

Definition 5.1 (Extremal position)**.** A finite scheme $\Gamma \subset A$ of degree $d \geq g+1$ is in *extremal position* if it is theta-general, and if it imposes only $g+1$ conditions on $|\mathcal{O}_A(2\Theta) \otimes \alpha|$ for general $\alpha \in A$.

Note that by upper semicontinuity of $h^i(\mathcal{I}_\Gamma(2\Theta)_\alpha)$, the number of conditions imposed by Γ is lower semicontinuous, so if for a general α it imposes $g+1$ conditions on $|\mathcal{O}_A(2\Theta) \otimes \alpha|$, it will impose at most $g+1$ conditions on $|\mathcal{O}_A(2\Theta) \otimes \alpha|$ for all α.

The following result give a description of the cohomological support locus of a finite scheme in extremal position in an irreducible ppav.

Lemma 5.2. *Let Γ be a scheme of degree $g+2$ in extremal position on A, and let $\Gamma_{g+1} \subsetneq \Gamma$ be any subscheme of degree $g+1$ in Γ. Then*

(i) *For all $\Gamma' \subset \Gamma_{g+1}$, such that $\deg \Gamma' = g$, $\mathcal{H}^{\Gamma',\Gamma_{g+1}}$ consists in one point, $\gamma_{\Gamma',\Gamma_{g+1}}$.*
(ii) *There exists $\alpha_{\Gamma_{g+1}}$ such that $\Theta_{\alpha_{\Gamma_{g+1}}}$ is the unique divisor contained in $V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta))$ and for every $\Gamma'' \subset \Gamma_{g+1}$ such that $\deg \Gamma'' = g$,*

$$\alpha_{\Gamma_{g+1}} = \gamma_{\Gamma'',\Gamma_{g+1}} + z_{\Gamma'',\Gamma_{g+1}}.$$

Proof. Given any subscheme $\Gamma' \subset \Gamma_{g+1}$ such that $\deg \Gamma' = g$, and for all $\gamma' \in \mathcal{H}^{\Gamma',\Gamma_{g+1}}$, we have that $\Gamma' \subset \Theta_{\gamma'}$ but, $\Gamma_{g+1} \not\subset \Theta_{\gamma'}$. Then by Lemma 3.6

$$\Gamma_{g+1} \subseteq \Theta_{\gamma'} + \Theta_\beta \Leftrightarrow \beta \in \Theta_{z_{\Gamma',\Gamma_{g+1}}}.$$

Imposing $\Gamma \not\subset \Theta_{\gamma'} + \Theta_\beta$ is a closed proper condition inside $\Theta_{z_{\Gamma',\Gamma_{g+1}}}$. Thus, for an open subset $U \subseteq \Theta_{z_{\Gamma',\Gamma_{g+1}}}$ if $\beta \in U$ then $\Gamma \not\subset \Theta_{\gamma'} + \Theta_\beta$. As Γ_{g+1} is theta-independent, and hence, imposes $g+1$ conditions on $|(2\Theta)_\alpha|$ for a general α, we have that $\gamma' + \beta$ could not be general, because if $\beta \in U$ there exists a divisor $\Theta_{\gamma'} + \Theta_\beta$ in $\left|(2\Theta)_{\gamma'+\beta}\right|$ that contain Γ_{g+1} and do not contain Γ, so if it was general Γ would impose $g+2$ conditions on $\left|(2\Theta)_{\gamma'+\beta}\right|$ which contradicts the extremality of Γ. Therefore,

$$\gamma' + \beta \in V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta))$$

so $U + \gamma'$ is contained in the subvariety $V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta))$. Since Θ is assumed to be irreducible, this gives

$$\Theta_{\gamma' + z_{\Gamma',\Gamma_{g+1}}} \subset V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta))$$

for any $\gamma' \in \mathcal{H}^{\Gamma',\Gamma_{g+1}}$ (given any subscheme $\Gamma' \subset \Gamma_{g+1}$ of degree g).

On the other hand, for any $\widetilde{\gamma}' \in \mathcal{H}^{\Gamma',\Gamma_{g+1}}$, if $\beta \notin \Theta_{z_{\Gamma',\Gamma_{g+1}}}$, the divisor $\Theta_{\widetilde{\gamma}'} + \Theta_\beta$ contains Γ' and do not contain Γ_{g+1} (cf. 3.6). Hence

$$B(\Gamma' \subset \Gamma_{g+1}, 2) \subseteq \Theta_{z_{\Gamma',\Gamma_{g+1}} + \widetilde{\gamma}'}.$$

In conclusion, it follows from (2) that

$$\Theta_{\gamma' + z_{\Gamma',\Gamma_{g+1}}} \subset V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) = B(\Gamma' \subset \Gamma_{g+1}, 2) \cup V(\mathcal{I}_{\Gamma'}(2\Theta)) \subset \Theta_{\widetilde{\gamma}' + z_{\Gamma',\Gamma_{g+1}}} \cup V(\mathcal{I}_{\Gamma'}(2\Theta))$$

Since by Lemma 4.4 $\dim V(\mathcal{I}_{\Gamma'}(2\Theta)) \leq g-2$, we get that $\gamma' = \widetilde{\gamma}'$, i.e. that $\mathcal{H}^{\Gamma',\Gamma_{g+1}}$ consists in one unique point that we call $\gamma_{\Gamma',\Gamma_{g+1}}$. Moreover $B(\mathcal{I}_{\Gamma'} \subset \Gamma_{g+1}, 2)$ is an open subset of $\Theta_{\gamma_{\Gamma',\Gamma_{g+1}} + z_{\Gamma',\Gamma_{g+1}}}$, $\overline{B(\mathcal{I}_{\Gamma'} \subset \Gamma_{g+1}, 2)} = \Theta_{\gamma_{\Gamma',\Gamma_{g+1}} + z_{\Gamma',\Gamma_{g+1}}}$ and $\Theta_{\gamma_{\Gamma',\Gamma_{g+1}} + z_{\Gamma',\Gamma_{g+1}}}$ is the unique divisor contained in $V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta))$. □

The first consequence of the previous lemma is that the finite schemes of degree $g+2$ in extremal position are nearly composed by curvilinear points. Indeed, we will prove later that they are actually composed by curvilinear points (cf. 6.8).

Corollary 5.3. *Let Γ be a finite scheme of degree $g+2$ in extremal position on A, then any subscheme of degree $g+1$ is composed by curvilinear points.*

Proof. Let Γ_{g+1} be any subscheme of Γ of degree $g+1$. We claim than Γ_{g+1} does not have two different subschemes Γ' and Γ'' of degree g and $z_{\Gamma',\Gamma_{g+1}} = z_{\Gamma'',\Gamma_{g+1}}$. We left to the reader to convince himself that this claim implies that Γ_{g+1} is composed by curvilinear points.
To prove the claim we argue by contradiction. Then by the previous Lemma

$$\gamma_{\Gamma',\Gamma_{g+1}} + z_{\Gamma',\Gamma_{g+1}} = \alpha_{\Gamma_{g+1}} = \gamma_{\Gamma'',\Gamma_{g+1}} + z_{\Gamma'',\Gamma_{g+1}}$$

and as Γ' and Γ'' have the same support this implies that

$$\gamma_{\Gamma',\Gamma_{g+1}} = \gamma_{\Gamma'',\Gamma_{g+1}}$$

But this is impossible because if Θ_α contains Γ' and Γ'' it must contain Γ_{g+1}. □

Remark 5.4. In particular if a 0-scheme Γ of degree $g+2$ is supported in at least two points, then it is composed by curvilinear points. So, a scheme of degree $g+2$ in extremal position falls in one of the following two cases:

(i) *Supported in a unique point.* Then it can be either a curvilinear point or a 0-scheme with two different curvilinear subschemes of degree $g+1$. A posteriori, we will see that this last possibility do not occur (Corollary 6.8).

(ii) *Supported in at least 2 points.* Then it is composed of curvilinear points.

6. An Extremal Finite Scheme Supported in a Unique Point

From now on we suppose, otherwise stated, that Γ is a finite scheme of degree $g+2$ in extremal position and only supported at the origin of the abelian variety. By Remark 5.4, there is a unique composition series, except that there could be two possible subschemes of degree $g+1$. We fix the two possible composition series for $\Gamma = \Gamma_{g+2}$,

$$\varnothing = \Gamma_0 \subsetneq \Gamma_1 = \{0\} \subsetneq \ldots \subsetneq \Gamma_g \subsetneq \begin{matrix} \Gamma'_{g+1} \\ \Gamma''_{g+1} \end{matrix} \subsetneq \Gamma_{g+2} = \Gamma.$$

To suppose that $\Gamma_1 = \{0\}$ simplifies the notation because, in this situation, $z_{\Gamma'',\Gamma'} = 0$ for all subschemes $\Gamma'' \subset \Gamma'$ of Γ.

From now on, to simplify some arguments in the proofs we will use Γ_{g+1}, when possible, to denote indistinctly Γ'_{g+1} or Γ''_{g+1}.

Lemma 6.1. *Let Γ as above, then for every $y \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g}$*

$$\Theta_y \cap V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \subseteq (\Theta_{\Gamma_2})_y \cup V(\mathcal{I}_{\Gamma_g}(2\Theta)), \tag{3}$$

where Γ_{g+1} is either Γ'_{g+1} or Γ''_{g+1}.

Proof. Consider $y \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g}$, i.e. a point y such that $\Gamma_{g-1} \subset \Theta_y$, but $\Gamma_g \not\subset \Theta_y$. Then if $\alpha \in \Theta_y$, we have that

$$\Gamma_g \subset \Theta_{\alpha-y} + \Theta_y \tag{4}$$

because we apply Lemma 3.6. Assume now that $\alpha \notin (\Theta_{\Gamma_2})_y$, then by Lemma 3.7, since Γ_{g+1} is a curvilinear point,

$$\Gamma_{g+1} \not\subset \Theta_{\alpha-y} + \Theta_y \tag{5}$$

Conditions (4) and (5) imply that $\alpha \notin B(\Gamma_g \subset \Gamma_{g+1}, 2)$. Now, if moreover, $\alpha \in V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) = B(\Gamma_g \subset \Gamma_{g+1}, 2) \cup V(\mathcal{I}_{\Gamma_g}(2\Theta))$ (cf. equality (2)), then we have $\alpha \in V(\mathcal{I}_{\Gamma_g}(2\Theta))$. □

Remark 6.2. Observe that we have not used the extremality condition. We have only used that Γ is a finite scheme of degree $g+2$ supported in the origin, such that every subscheme of degree $g+1$ is curvilinear.

The following result specify the structure of the cohomological support locus for an extremal finite scheme supported in a unique point.

Corollary 6.3. *Let Γ as above, then*

$$V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) = \Theta_\gamma \cup R,$$

where R has at most dimension 1.

Proof. By the previous Lemma 6.1 we have that

$$V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \cap \Theta_y = \left((\Theta_{\Gamma_2})_y \cap V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta))\right) \cup \left(V(\mathcal{I}_{\Gamma_g}(2\Theta)) \cap \Theta_y\right).$$

Now, recall that by Lemma 5.2 $V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) = \Theta_\gamma \cup X$ where, X is of codimension greater than one and is included in $V(\mathcal{I}_{\Gamma_g}(2\Theta))$. So, intersecting the previous equality by Θ_γ we obtain

$$\Theta_\gamma \cap \Theta_y = \left((\Theta_{\Gamma_2})_y \cap \Theta_\gamma\right) \cup \left(\Theta_\gamma \cap \Theta_y \cap V(\mathcal{I}_{\Gamma_g}(2\Theta))\right).$$

As $\Theta_\gamma \cap \Theta_y$ has pure codimension 2 and Θ is an ample divisor, we must have $V(\mathcal{I}_{\Gamma_g}(2\Theta)) - R \subseteq \Theta_\gamma \cap \Theta_y$ where R is at most of dimension 1. Since $X \subseteq V(\mathcal{I}_{\Gamma_g}(2\Theta))$ we have that $X \subseteq \Theta_\gamma$ and

$$V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) = \Theta_\gamma \cup R.$$

□

The following result is the key result that allows us to construct a unidimensional family of degenerate trisecants.

Lemma 6.4. *Let Γ be a finite scheme of degree $g+2$ as above. Then for every $y \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g}$*

$$\Theta_y \cap \Theta_\gamma \subseteq (\Theta_{\Gamma_2})_y \cup (\Theta_{\Gamma_2})_\gamma\,, \tag{6}$$

where γ is either $\gamma_{\Gamma_g,\Gamma'_{g+1}}$ or $\gamma_{\Gamma_g,\Gamma''_{g+1}}$.

Proof. Fix $y \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g}$.
Observe that Θ_{Γ_2} is of pure codimension 2. Then, we define N as the union of the irreducible components of Θ_{Γ_2} such that $N_y \subseteq \Theta_\gamma$.

Claim 1: $\Theta_\gamma \cap \Theta_y = N_y \cup F$, where F the fixed part of the algebraic system $\Theta_\gamma \cap \Theta_y$.
By Lemma 6.1 we have that

$$V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \cap \Theta_y = \big((\Theta_{\Gamma_2})_y \cap V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta))\big) \cup \big(V(\mathcal{I}_{\Gamma_g}(2\Theta)) \cap \Theta_y\big)$$

Now, consider the components of codimension 2 on both sides of the equality. By previous Lemma and definition of N,

$$\Theta_\gamma \cap \Theta_y = N_y \cup F, \tag{7}$$

where F is the fixed part of the algebraic system $\Theta_\gamma \cap \Theta_y$.

Remark 6.5. Observe that, $\Theta_\gamma \cap \Theta_y \not\subseteq V(\mathcal{I}_{\Gamma_g}(2\Theta))$, because the intersection of the two theta divisors is of codimension 2 and y moves in space of (at least) dimension 1. In fact, one should prove that the intersection is not actually fixed, so it really moves. But if $\Theta_\gamma \cap \Theta_y$ were fixed then $\Theta_\gamma \cap \Theta_{y_1} \subseteq \Theta_\gamma \cap \Theta_{y_2} \subseteq \Theta_{y_2}$ for $y_1 \neq y_2 \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g}$ and this is impossible. In particular $N \neq \varnothing$.

Claim 2: $\Theta_\gamma \cap \Theta_y \subseteq (\Theta_{\Gamma_2})_\gamma \cup (\Theta_{\Gamma_2})_y$.
Translating (7) by $-\gamma - y$ we obtain

$$\Theta_{-y} \cap \Theta_{-\gamma} = N_{-\gamma} \cup F_{-y-\gamma}$$

As Θ is symmetric the left hand side is $(-1)^*\left(\Theta_y \cap \Theta_\gamma\right)$ so replacing this in the equality (7) we obtain that, $(-1)^*\left(N_y \cup F\right) = N_{-\gamma} \cup (F)_{-y-\gamma}$ or, in other words,

$$N_y \cup F = ((-1)^*N)_\gamma \cup ((-1)^*F)_{y+\gamma}.$$

As F does not move when y varies in a component of $\mathcal{H}^{\Gamma_{g-1},\Gamma_g}$, then $F = ((-1)^*N)_\gamma$ (in particular $F \neq \varnothing$), and

$$\Theta_\gamma \cap \Theta_y = N_y \cup ((-1)^*N)_\gamma\,.$$

Recall that $N \subset \Theta_{\Gamma_2}$. Then from this equality we deduce that $\Theta_\gamma \cap \Theta_y \subseteq (\Theta_{\Gamma_2})_y \cup ((-1)^*\Theta_{\Gamma_2})_\gamma$ and noting that $(-1)^*\Theta_{\Gamma_2} = \Theta_{\Gamma_2}$ because Θ is symmetric and Γ_2 is supported at the origin, we infer the claim. □

The following little lemma will allow us to give a lower bound for the dimension of Θ_{Γ_3}. This control on the dimension of Θ_{Γ_3} will be crucial in assuring that the finite scheme in extremal position is included in a unique Abel-Jacobi curve.

Lemma 6.6. *Let Γ be a finite scheme of degree $g+2$ as above. Then for every $y \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g}$,*

$$\Theta_{\Gamma_2} \cap B(\Gamma_{g+1} \subset \Gamma_{g+2}, 2)_{-y} \subseteq \Theta_{\Gamma_3},$$

where Γ_{g+1} is either Γ'_{g+1} or Γ''_{g+1}.

Proof. If $\alpha \in (\Theta_{\Gamma_2})_y$, then by Lemma 3.7, $\Gamma_{g+1} \subset \Theta_{\alpha-y} + \Theta_y$. Hence, if we define

$$U = \Theta_{\Gamma_2} - \Theta_{\Gamma_3},$$

then $\alpha - y \in U \Rightarrow \alpha \notin B(\Gamma_{g+1} \subseteq \Gamma_{g+2}, 2)$ (Lemma 3.7).
So $B(\Gamma_{g+1} \subset \Gamma_{g+2}, 2)_{-y} \subset A - U$ and, therefore, $\Theta_{\Gamma_2} \cap B(\Gamma_{g+1} \subset \Gamma_{g+2}, 2)_{-y} \subseteq \Theta_{\Gamma_2} - U = \Theta_{\Gamma_3}$. □

Now, we are ready to prove the main result for finite schemes in extremal position supported in a unique point.

Proposition 6.7. *Let (A, Θ) be an irreducible principally polarized abelian variety of dimension $g > 3$, and let $\Gamma \subset A$ be a finite scheme of degree $g+2$, supported in a unique point and in extremal position.*
Then (A, Θ) is the canonically polarized Jacobian of a curve C and $\Gamma \subset C$ for a unique Abel-Jacobi embedding $C \hookrightarrow J(C)$.

Proof. By a well-known result (see for example [Ma, Proposition 1]), the inclusion (6), $\Theta_y \cap \Theta_\gamma \subseteq (\Theta_{\Gamma_2})_y \cup (\Theta_{\Gamma_2})_\gamma$, is equivalent to the existence of an inflectionary trisecant at the point $\frac{1}{2}(y-\gamma)$ (here the factor $\frac{1}{2}$ denotes the counterimage by the isogeny multiplication by 2). In other words, the inclusion (6) implies the existence of a finite subscheme Y of degree 3, supported in the origin and independent of $y - \gamma$, such that $\frac{1}{2}(y-\gamma) + Y \subset \psi^{-1}(l)$ for some line $l \subset \mathbb{P}^N$, where $\psi : X \to \mathbb{P}^N = |2\Theta|^*$ is the Kummer morphism. Following the proof of Marini, it is clear that $\Gamma_2 \subset Y$ (see also [W1, Proposition 2.14]).

Hence, taking $u := \frac{1}{2}(y-\gamma)$

$$\left(\mathcal{H}^{\Gamma_{g-1},\Gamma_g}\right)_{-\gamma} \subseteq V = \left\{2u \mid u + Y \subset \psi^{-1}(l) \text{ for some line } l \in \mathbb{P}^N\right\}.$$

Recall that by Remark 3.10, $\dim \mathcal{H}^{\Gamma_{g-1},\Gamma_g} \geq 1$ so by the Gunning-Welters criterion [W2, Theorem 0.5], V is a smooth irreducible curve and (A, Θ) is the polarized jacobian of V. Moreover by [W2, Proposition 2.14], we know that $Y \subset V = \overline{\left(\mathcal{H}^{\Gamma_{g-1},\Gamma_g}\right)_{-\gamma}}$.

To prove that Γ is contained in a unique Abel-Jacobi curve we observe first that Γ_3 is contained in an Abel-Jacobi curve C. This is a direct consequence of the Lemma 6.6, $\Theta_{\Gamma_2} \cap B(\Gamma_{g+1} \subset \Gamma_{g+2}, 2)_{-y} \subseteq \Theta_{\Gamma_3}$, once we know that $\Theta_{\Gamma_2} \cap B(\Gamma_{g+1} \subset \Gamma_{g+2}, 2)_{-y} \neq \varnothing$ (we will see this at the end of the proof). Therefore, we have that Θ_{Γ_3} is of codimension 2, so it contains the locus of translates that contain the curve. Indeed, if Γ_2 is included in an Abel-Jacobi curve, $\Theta_{\Gamma_2} = W \cup H$ where $W = \{\alpha \mid C \subset \Theta_\alpha\}$ and $H = \overline{\{\alpha \mid \Gamma_2 \subset \Theta_\alpha \text{ and } C \not\subset \Theta_\alpha\}}$ are irreducible of codimension 2. If Γ_3 is not contained in any Abel-Jacobi curve, then $\Theta_{\Gamma_3} = W' \cup H'$ where W' and H' are proper subsets of W and H, because the intersection of all the theta translates that contain the Abel-Jacobi curve is the curve.

Now suppose that $\Gamma_i \subset C$, but $\Gamma_{i+1} \not\subseteq C$ $(i \geq 3)$. In this case, $\Theta_{\Gamma_i} = W \sqcup H_i$, where $W = \{\alpha \in A \mid C \subset \Theta_\alpha\}$. We know that W is irreducible of dimension $g-2$. We have supposed that $\Gamma_{i+1} \not\subseteq C$, hence if $\Theta_{\Gamma_{i+1}} = W_{i+1} \sqcup H_{i+1}$ where $W_{i+1} = W \cap \Theta_{\Gamma_i}$, W_{i+1} has codimension 1 in W. Therefore $\dim \mathcal{H}^{\Gamma_i,\Gamma_{i+1}} = g-2$ (because it contains $W - W_{i+1}$).
Then $\Theta_{\Gamma_{i+2}} = W_{i+2} \sqcup H_{i+2}$, and W_{i+2} is an open set of codimension at most 1 in W_{i+1}. Hence $\dim \mathcal{H}^{\Gamma_{i+1},\Gamma_{i+2}} \geq g-3$.
Repeating this process we infer that $\dim \mathcal{H}^{\Gamma_g,\Gamma_{g+1}} \geq i-2$. But we have seen that $\Gamma_3 \subset C$, so $i \geq 3$ and we knew that $\mathcal{H}^{\Gamma_g,\Gamma_{g+1}} = \{\gamma\}$, so we get a contradiction.

Finally, it remains to show that $\Theta_{\Gamma_2} \cap B(\Gamma_{g+1} \subset \Gamma_{g+2}, 2)_{-y} \neq \varnothing$. But if this intersection were empty, then $\Theta_{\Gamma_2} \subset V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta))_{-y}$. So, by Corollary 6.3, $\Theta_{\Gamma_2} \subset \Theta_{y-\gamma}$ for every $y \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g}$. This is an absurd because says that any α such that $\Gamma_2 \subset \Theta_\alpha$, satisfies $V = \overline{\left(\mathcal{H}^{\Gamma_{g-1},\Gamma_g}\right)}_{-\gamma} \subset \Theta_\alpha$!!! □

Observe that a priori, in the previous proof, we have obtained two different curves $V = \overline{\left(\mathcal{H}^{\Gamma_{g-1},\Gamma_g}\right)}_{-\gamma}$, given by the two possible values of γ. But, afterwards, we have proved that Γ is contained in any of the two curves. Hence, a posteriori, we can affirm that $\Gamma'_{g+1} = \Gamma''_{g+1}$ and consequently that the curve V is unique. Therefore, improving Corollary 5.3 we obtain,

Corollary 6.8. *Any scheme Γ of degree $g+2$ in extremal position on A is composed by curvilinear points.*

7. The General Case

The goal of this section is to complete the proof of Theorem A. We will see how we have to choose the composition series to apply the same arguments of a unique curvilinear point. We will use the trick of considering partially two different decomposition series (Γ_i and $\widetilde{\Gamma}_i$).

Hence, the degenerate trisecant case, that seems so particular is the most general one. Some similar behavior has been observed by Marini ([Ma]) and Debarre ([De]), i.e. that the existence of a non-degenerate trisecant line implies the existence of a degenerate trisecant.

Proof of Theorem A. We will prove that we have the same ingredients that allows us to prove the case of a unique curvilinear point (cf. Proposition 6.7).

Since we have proved the case of a finite scheme supported on a unique point in Proposition 6.7, we assume that $\Gamma = \Gamma_{g+2}$ is supported in at least two points. The case of distinct reduced points has been proved by G. Pareschi and M. Popa (cf. [PP2]), hence, we also assume that the finite scheme $\Gamma = \Gamma_{g+2}$ is non-reduced. Then, at least, one of the irreducible components is non-reduced, and suppose this irreducible component is supported at 0. Call $\widetilde{\Gamma}_2 \subset \Gamma$ the non-reduced subscheme of degree 2 supported at 0. Then we can choose a composition series for Γ

$$\varnothing = \Gamma_0 \subsetneq \Gamma_1 = \{p\} \subsetneq \ldots \subsetneq \Gamma_g \subsetneq \Gamma_{g+1} \subsetneq \Gamma_{g+2} = \Gamma,$$

where possibly $\Gamma_2 \neq \widetilde{\Gamma}_2$, but such that,

$$z_{\Gamma_{g-1},\Gamma_g} = z_{\Gamma_g,\Gamma_{g+1}} = 0 \neq z_{\Gamma_{g+1},\Gamma_{g+2}}.$$

We remark, that we are not assuming that $\widetilde{\Gamma}_2 \subseteq \Gamma_g$. In fact, we can only assure that $\widetilde{\Gamma}_2 \subseteq \Gamma_{g+1}$.

Example 7.1. We give two opposite examples.

(i) Consider Γ_{g+2} consisting in g distinct points $\{p_1, \ldots, p_g\}$ and a non-reduced point of degree 2, $Y \ni 0$. In this case we choose $\widetilde{\Gamma}_2 = Y$, $\Gamma_{g+1} = \{p_2, \ldots, p_g\} \cup Y$, $\Gamma_g = \{p_2, \ldots, p_g\} \cup \{0\}$ and $\Gamma_{g-1} = \{p_2, \ldots, p_g\}$. Observe that here $\Gamma_2 \neq \widetilde{\Gamma}_2$.

(ii) Consider Γ_{g+2} consisting in one reduced point p and a curvilinear point of degree $g+1$, $Y = Y_{g+1} \supset Y_g \supset \ldots \supset Y_1 = \{0\}$. In this case we choose $\widetilde{\Gamma}_2 = Y_2$, $\Gamma_{g+1} = Y_{g+1}$, $\Gamma_g = Y_g$ and $\Gamma_{g-1} = Y_{g-1}$. Here $\Gamma_2 = \widetilde{\Gamma}_2$.

In fact, we note that the first example is the most delicate one.

Then for every $y \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g}$, if $\alpha \in \Theta_y$, we have that

$$\Gamma_g \subset \Theta_{\alpha-y} + \Theta_y$$

because $z_{\Gamma_{g-1},\Gamma_g} = 0$ (cf. Lemma 3.6). Assume moreover that $\alpha \notin \left(\Theta_{\widetilde{\Gamma}_2}\right)_y$, then, by Lemma 3.7, since $z_{\Gamma_g,\Gamma_{g+1}}$ is also 0 and $\widetilde{\Gamma}_2$ is the non-reduced scheme of degree 2 supported in the origin,

$$\Gamma_{g+1} \not\subset \Theta_{\alpha-y} + \Theta_y$$

From this two inclusions we deduce that $\alpha \notin B(\Gamma_g \subset \Gamma_{g+1}, 2)$. Now, if moreover, $\alpha \in V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) = B(\Gamma_g \subset \Gamma_{g+1}, 2) \cup V(\mathcal{I}_{\Gamma_g}(2\Theta))$ (cf. 2), then $\alpha \in V(\mathcal{I}_{\Gamma_g}(2\Theta))$. Hence, we have proved that in this situation

$$\Theta_y \cap V(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \subseteq \left(\Theta_{\widetilde{\Gamma}_2}\right)_y \cup V(\mathcal{I}_{\Gamma_g}(2\Theta)),$$

This inclusion substitutes the previous (3) and allows us to reprove Lemma 6.4. More precisely, with the previous decomposition series, for every $y \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g}$

$$\Theta_y \cap \Theta_\gamma \subseteq \left(\Theta_{\widetilde{\Gamma}_2}\right)_y \cup \left(\Theta_{\widetilde{\Gamma}_2}\right)_\gamma,$$

where γ is $\gamma_{\Gamma_g,\Gamma_{g+1}}$. Therefore, also in this case, we have enough degenerate trisecants to ensure that we are in a Jacobian by the Gunning-Welters criterion.

To prove that Γ_{g+2} is contained in an Abel-Jacobi curve, we can go over again the previous argument (Lemma 6.6), once we choose $\widetilde{\Gamma}_3$ appropriately. Since we have assumed that $z_{\Gamma_{g+1},\Gamma_{g+2}} \neq 0$, we choose $\widetilde{\Gamma}_3 = \widetilde{\Gamma}_2 \cup \left\{z_{\Gamma_{g+1},\Gamma_{g+2}}\right\}$. Then, the following property holds

$$\left(y \in \mathcal{H}^{\Gamma_{g-1},\Gamma_g} \text{ implies } \Gamma_{g+2} \subset \Theta_y \cup \Theta_{\alpha-y}\right) \text{ if, and only if, } \left(\widetilde{\Gamma}_3 \in \Theta_\alpha\right).$$

Example 7.2. In the previous examples.

(i) $\widetilde{\Gamma}_3 = Y \cup \{p_g\}$.

(ii) $\widetilde{\Gamma}_3 = Y_2 \cup \{p\}$.

Now we have redefined Γ_2 and Γ_3, the proof of Proposition 6.7, word by word, (changing Γ_2 by $\widetilde{\Gamma}_2$ and Γ_3 by $\widetilde{\Gamma}_3$) prove Theorem A for the non-reduced case. □

Remark 7.3. The trick of considering this two decomposition series allows us to show all the non-reduced cases are analogous to the case in Proposition 6.7 and that there exist enough totally degenerated trisecants. However, Theorem A could be proved showing that if Γ_{g+2} is supported in at least three points, there exists enough honest trisecants, and that if it is non-reduced and supported in at least two points there exist enough partially degenerate trisecants (i.e. a line tangent to the Kummer variety at some point and that meets the Kummer variety at some different point).

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